FROM FIELDS TO TOPOLOGY
Constructing TQFT invariants through Physics

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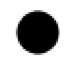
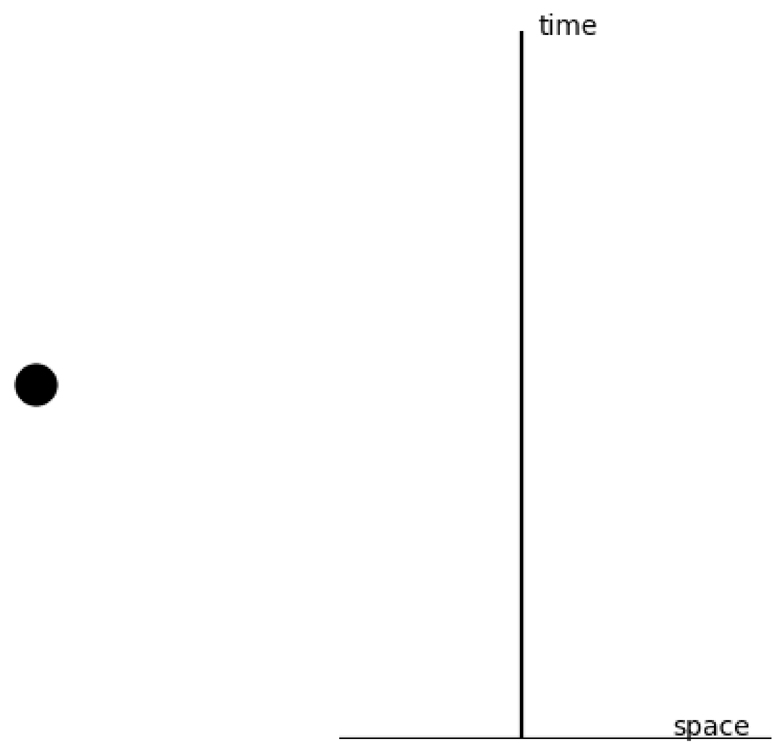
Field Theory

Traditionally, Physics has been interested in describing the evolution of a particle's position in time (i.e., its dynamics) through some equations of motion

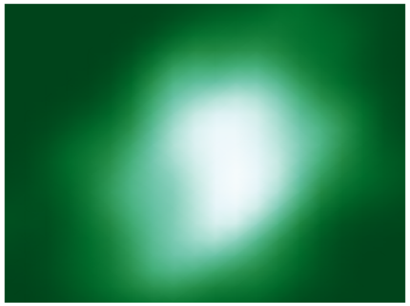


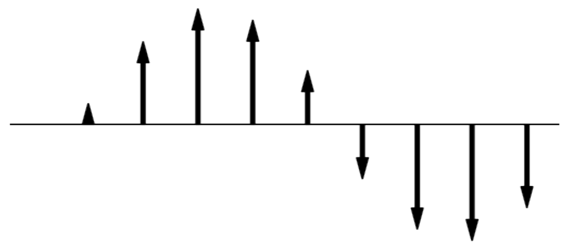
We can see the equations of motion as describing a (smooth) trajectory through space-time, $\mathbb{R}^3 \times \mathbb{R} = \mathbb{R}^4$

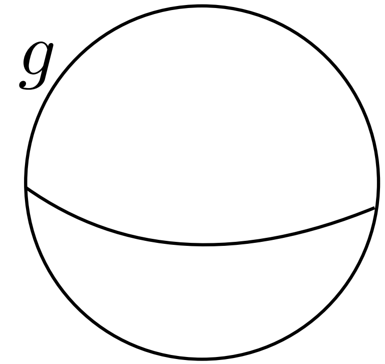


Field Theory

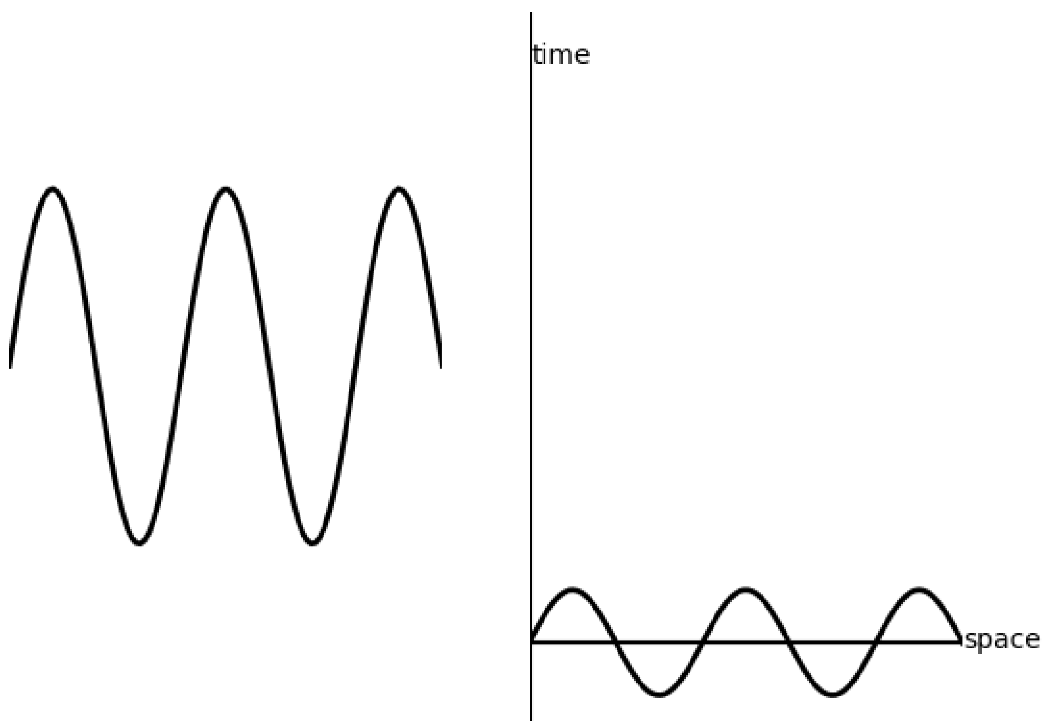
Tensor **fields** in space $\mathbb{R}^3$ may also change through time







Field theory assigns equations of motion defining the dynamics of fields



The evolution of the field is enconded in the **action functional**, $S[A, \partial A]$, such that, at all $\vec{x}, t$, the value of the field minimizes the action

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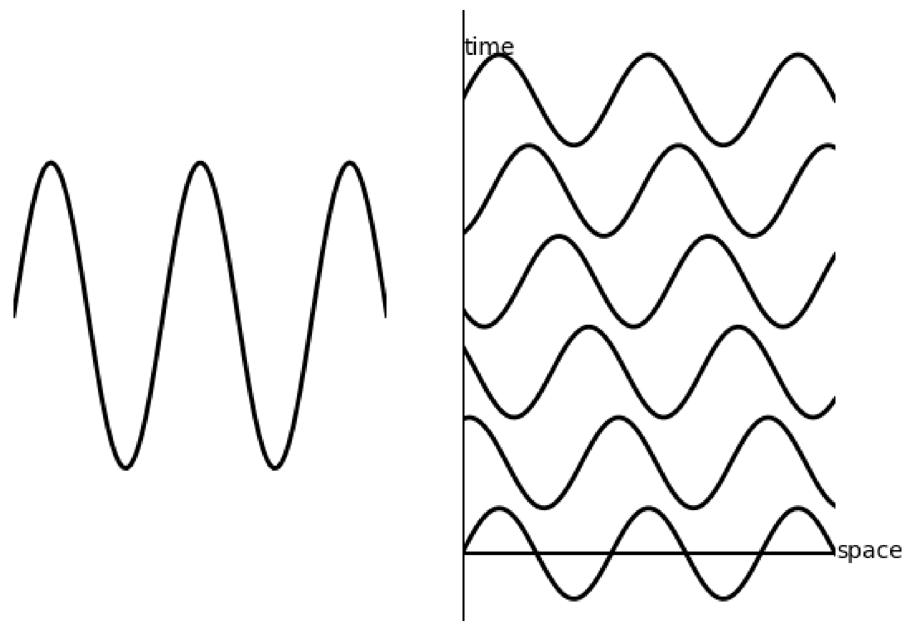
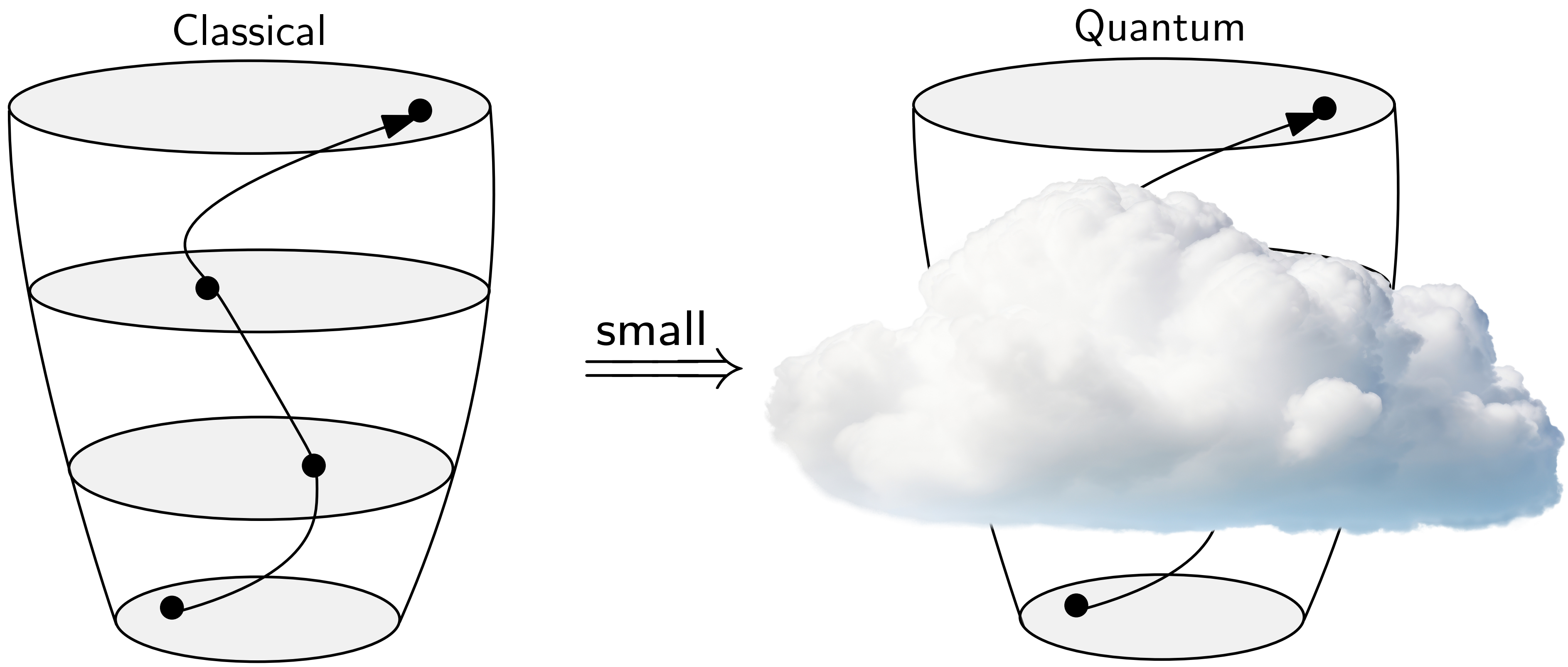
Example: Consider the field of scalars $A(x,t)$ with $A(x,0) = \sin(x)$ and with action $S = \int \left(\frac{\partial}{\partial x} A\right)^2 - \left(\frac{\partial}{\partial t} A\right)^2 \, dx \, dt$. Then the minimizer of the action, $A(x,t)$, is as



Once we move to the quantum realm, we cannot deterministically "follow" particles through space-time anymore

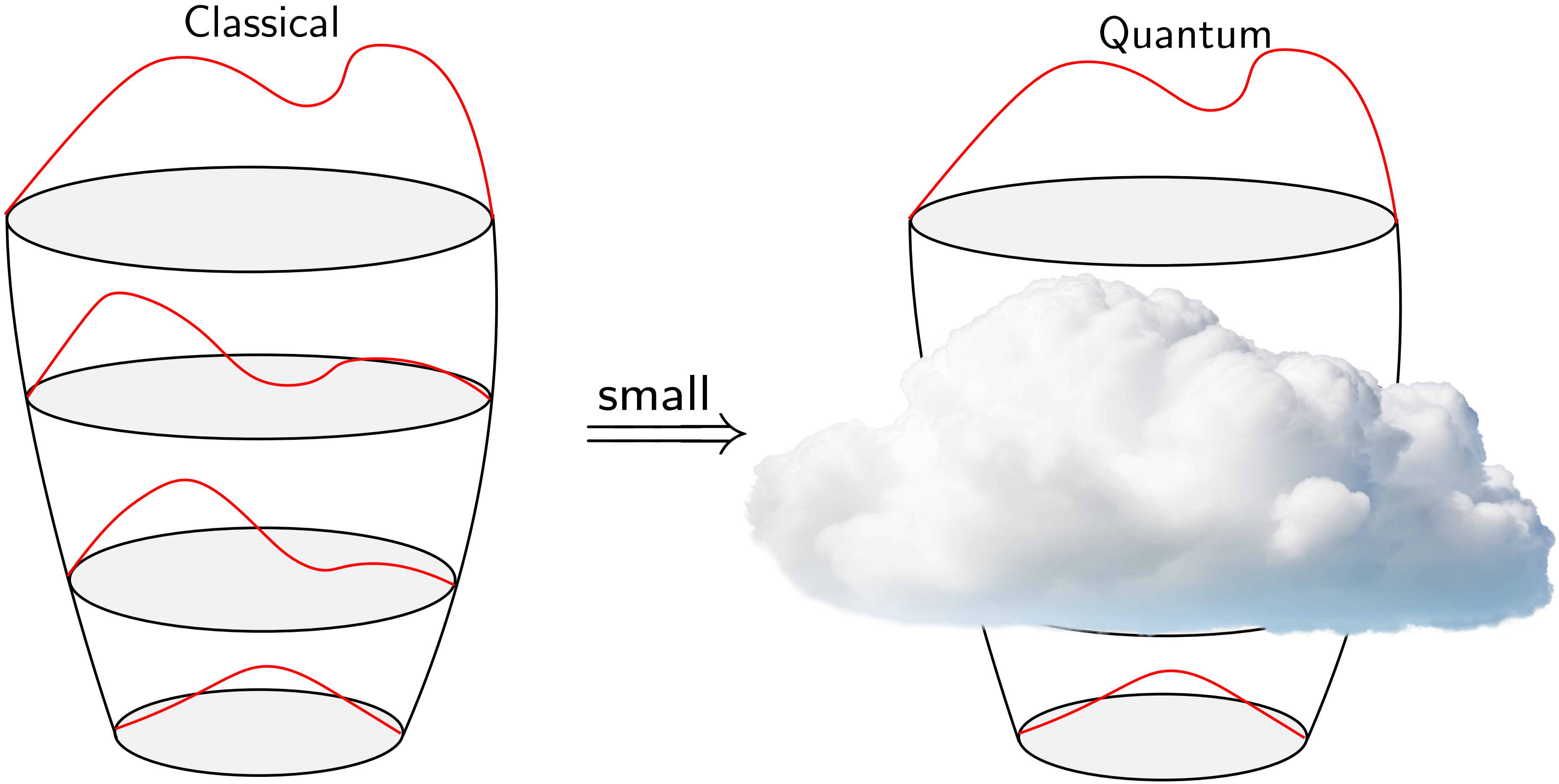


We can only talk about the **probability** of the particle, leaving from the position $\vec{x}_0$, to reach some position $\vec{x}$ at time t

$$\text{prob}_{\vec{x_0}}(\vec{x}, t)$$

Topological Quantum Field Theory

The same idea works for fields, that is we only know them at the beginning and at the end of the experiment



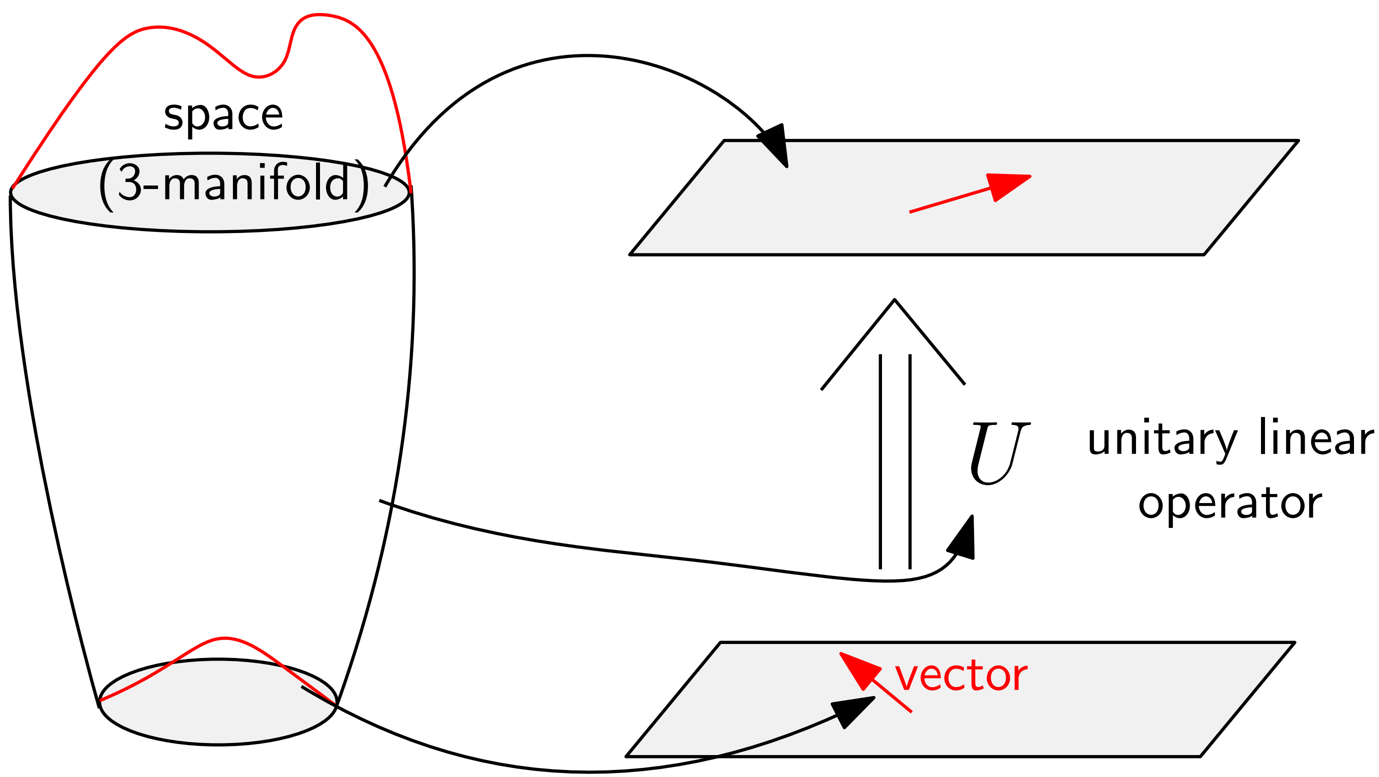
But we may similarly ask for the probability of an initial field A_0 to become A after some time t

$$\text{prob}_{A_0}(A, t)$$

Quantum mechanics describes this probability by linear algebra

- associate space $\mathbb{R}^3$ to some Hilbert space $H, \langle \cdot, \cdot \rangle$
- associate the field A_0 to a vector v_0 in H
- associate the field A to a vector v in H
- there is a unitary operator $U : H \to H$ such that

$$\text{prob}_{A_0}(A, t) = |\langle v, U v_0 \rangle|^2$$



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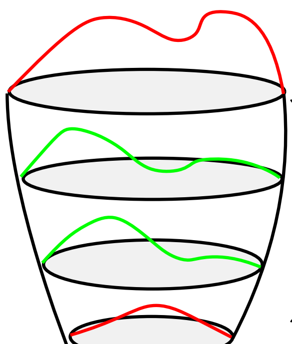
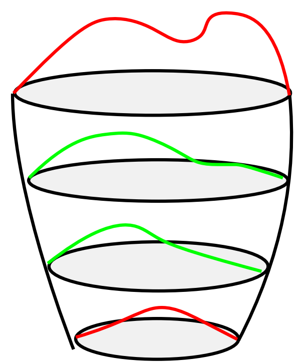
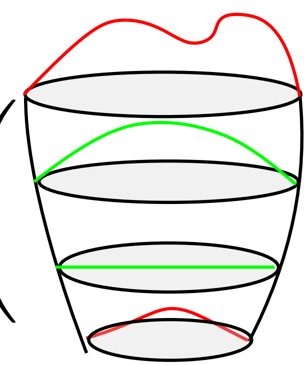
- the operator U is given by "adding" the actions evaluated at all possible fields from t_0 to t

U "=" $S\Big($  $\Big) + S\Big($  $\Big) + S\Big($  $\Big) + \ldots$

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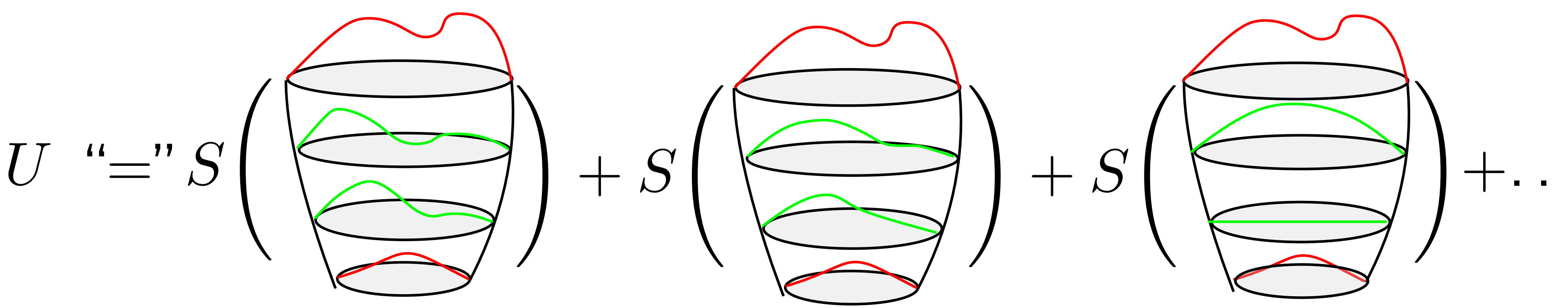
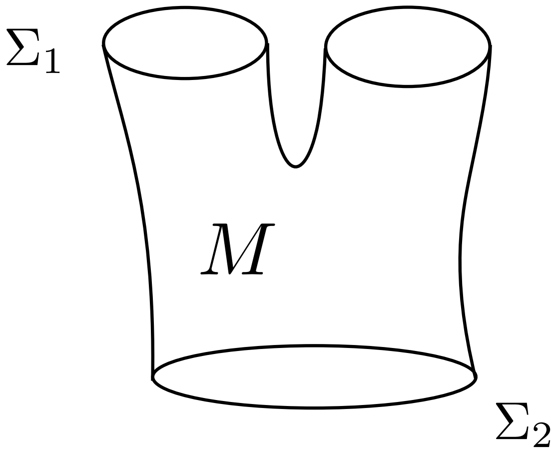


Once we fix S, the linear operator U depends solely on the geometry/topology of the ambient manifold of space-time M

A d-**cobordism** is a triple $(M; \Sigma_1, \Sigma_2)$ consisting of a d-dimensional compact manifold M, with two closed $d-1$-dimensional manifolds for boundary, Σ_1, Σ_2



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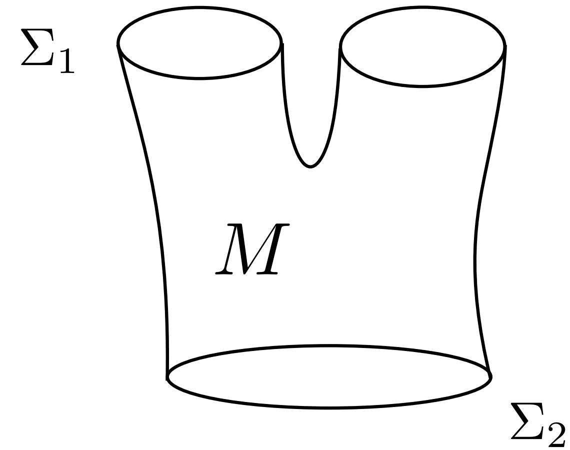
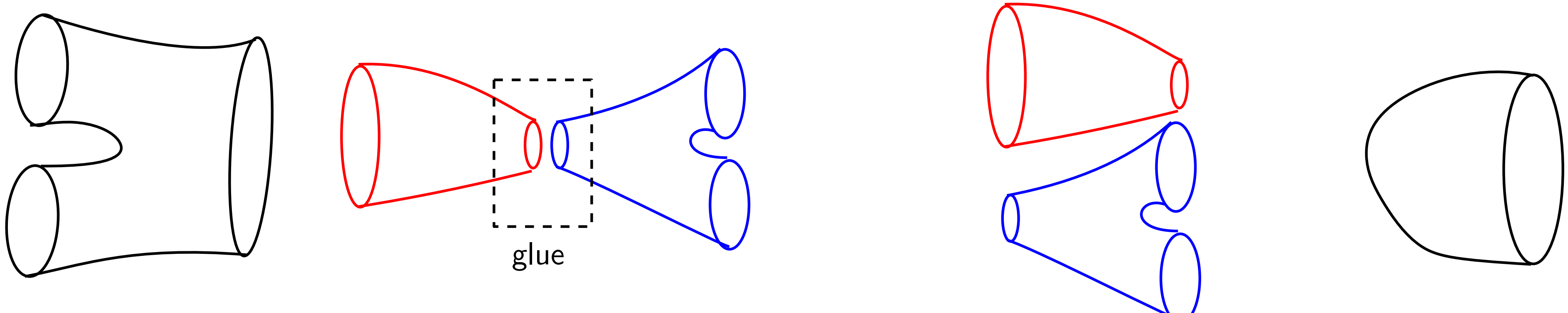


The class of all d-cobordisms (up to orientation preserving diffeomorphisms*) forms a monoidal category

$$(d\mathbf{Cob},\circ,\sqcup,\emptyset)$$

$\Sigma_1 \xrightarrow{M} \Sigma_2$ $\quad$ $\Sigma_1 \xrightarrow{M_1} \Sigma_2 \circ \Sigma_2 \xrightarrow{M_2} \Sigma_3$ $\quad$ $\Sigma_1 \sqcup \Sigma_2 \xrightarrow{M_1 \sqcup M_2} \Sigma_3 \sqcup \Sigma_4$ $\quad$ $\emptyset \xrightarrow{M} \Sigma_1$



*The usual definition of the category d**Cob** usually does not include the quotient by diffeomorphisms, but this means changing a little the definition of TQFTs

Topological Quantum Field Theory

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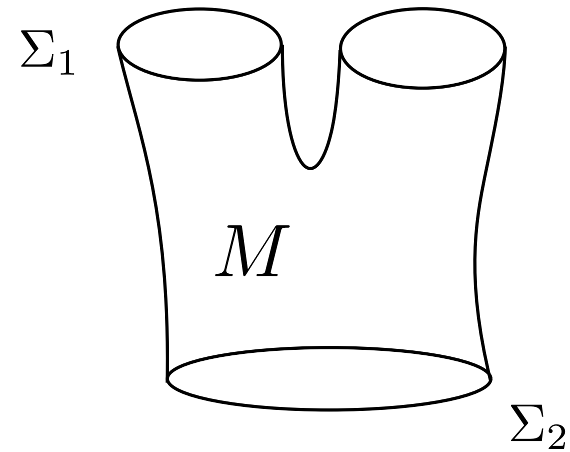
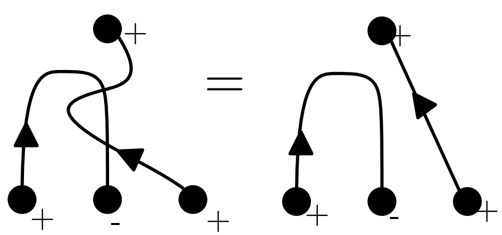


The class of all d-cobordisms (up to orientation preserving diffeomorphisms) forms a monoidal category

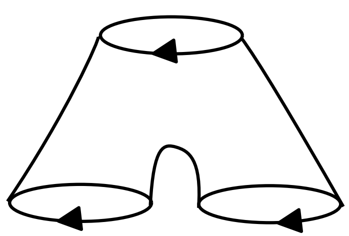
$$(d\mathbf{Cob},\circ,\sqcup,\emptyset)$$

Ex.:

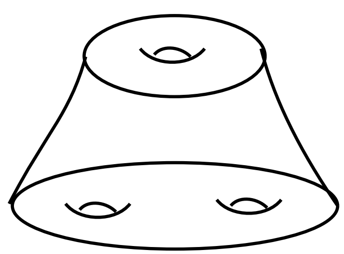
1Cob



2Cob



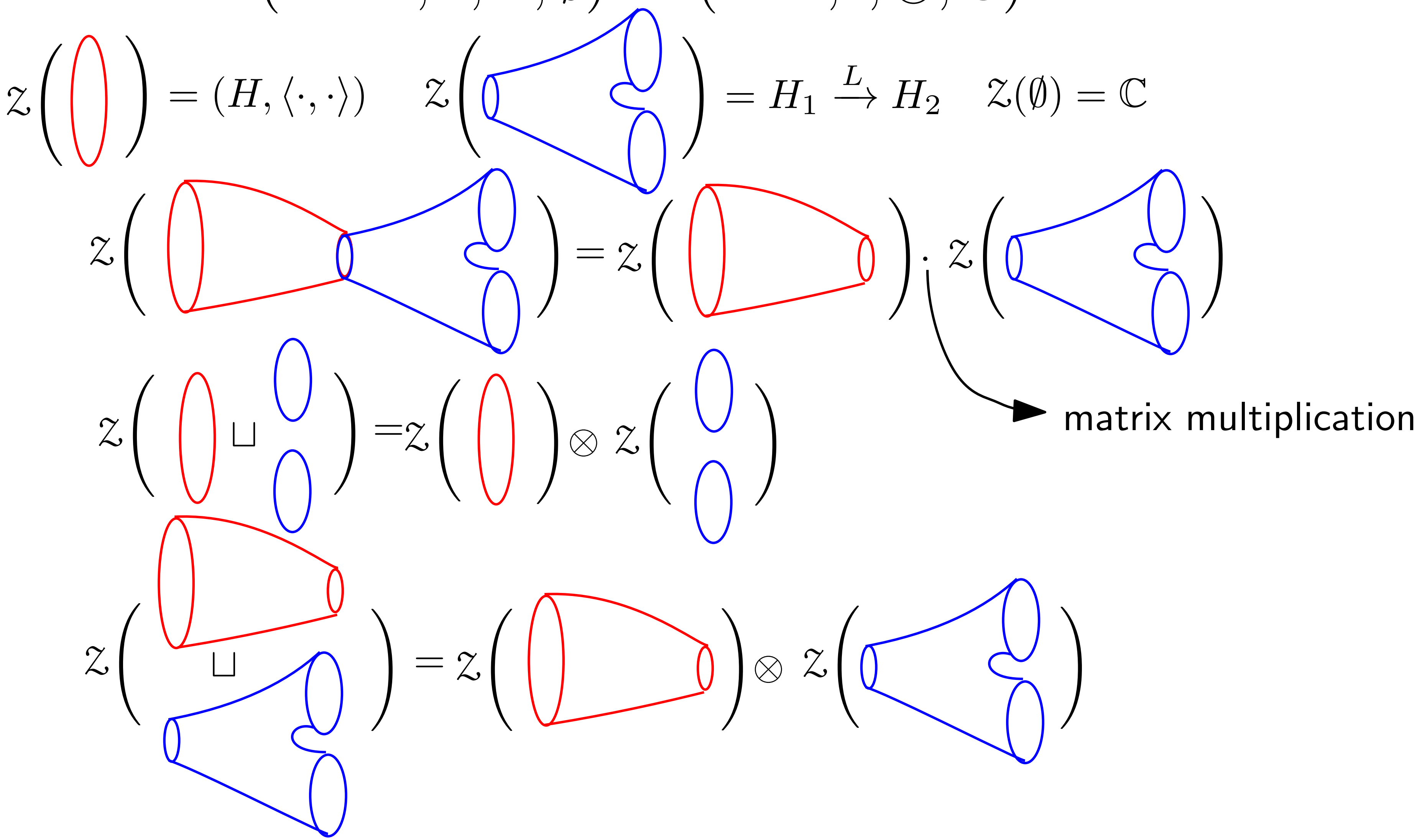
3Cob



Topological Quantum Field Theory

A d-dimensional Topological Quantum Field Theory (TQFT)* is a monoidal functor

$$\mathcal{Z} : (d\textbf{Cob}, \circ, \sqcup, \emptyset) \to (\textbf{Hilb}, \cdot, \otimes, \mathbb{C})$$



* The usual definition of TQFT is a little bit less restrictive, but we again do not care

Topological Quantum Field Theory

Example: 1-TQFTs are in one-to-one relation with finite dimensional Hilbert spaces

$$\mathcal{Z}(\bullet_+) = V \quad \mathcal{Z}(\bullet_-) = V^* \quad \mathcal{Z}(\uparrow) = \mathsf{id}_V \quad \mathcal{Z}(\downarrow) = \mathsf{id}_{V^*}$$

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Example:4-TQFTs will be of the exact form that Physicists use

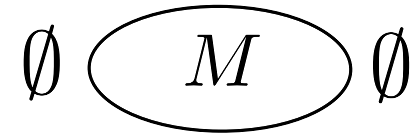
$$\text{space } \mathbb{R}^3 \xrightarrow{\mathcal{Z}} \text{Hilbert space}$$

$$\text{space-time } \mathbb{R}^4 \xrightarrow{\mathcal{Z}} \text{linear operator}$$

$$S \text{ fixed } \mathcal{Z}$$

Invariants of Topological Quantum Field Theory

A compact **closed** d-dimensional manifold can be seen as cobordism $\emptyset \xrightarrow{M} \emptyset$



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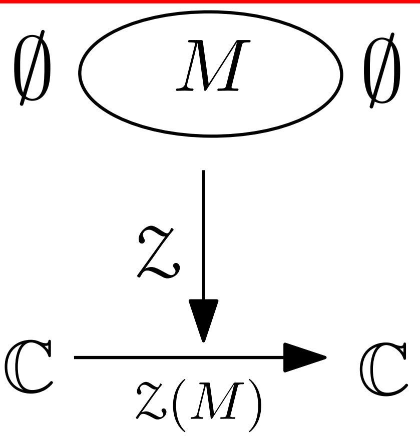

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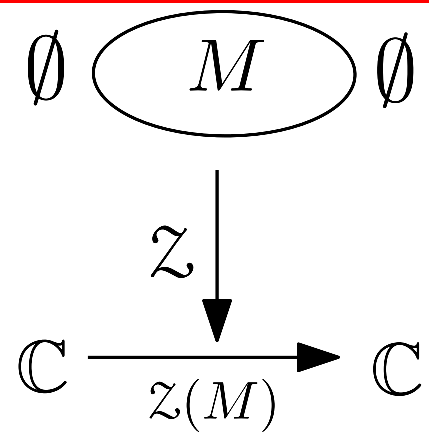

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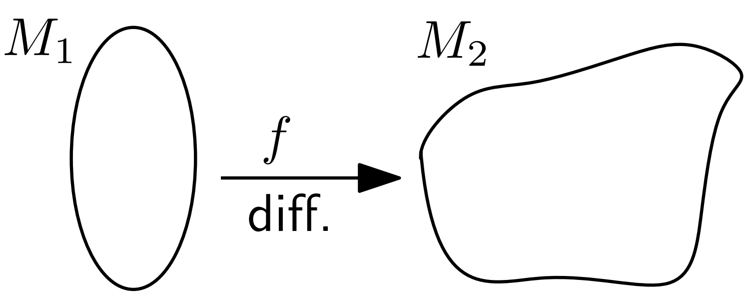
Because elements of d**Cob** are defined up to diffeomorphisms, the scalar **depends only on the diffeomorphism type** of M



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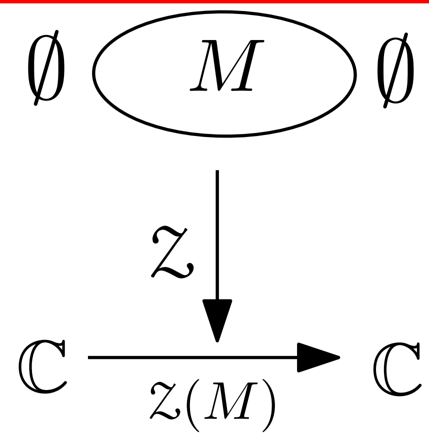

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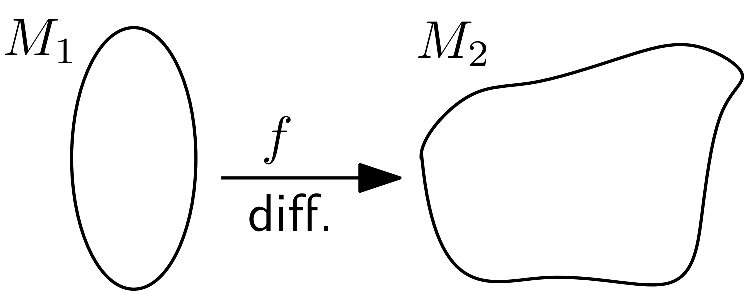

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d-dimensional TQFT $\Longrightarrow$ invariant for d-dimensional compact closed manifolds

Topological Quantum Field Theory

Example: 1-TQFTs are in one-to-one relations with finite dimensional Hilbert spaces and the induced invariant on S^1 always equals 1

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Example: from 3-TQFT on, things become *much* more complicated, but TQFT invariants still have received plenty of attetion (e.g., Witten-Reshetikhin-Turaev invariants, etc)

Computing TQFTs: what gives?

Fix an object Σ in d**Cob** (i.e., a $(d-1)$-dimensional closed manifold) and a diffeomorphism $f : \Sigma \to \Sigma$. Then there is a cobordism $\Sigma \xrightarrow{M} \Sigma$ given by glueing one end with the identity and the other with f

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Define the **mapping class group** of a closed $(n-1)$

$$\mathrm{Mod}(\Sigma) = \mathrm{Diff}^+(\Sigma)/\mathrm{isotopy}$$

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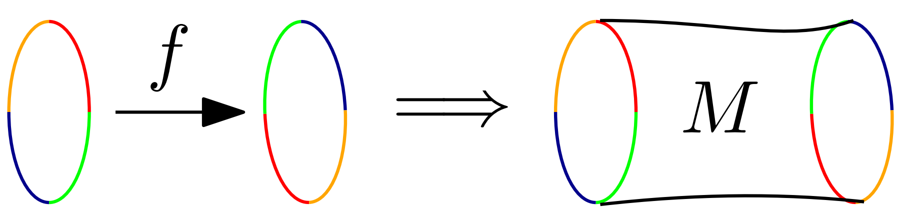
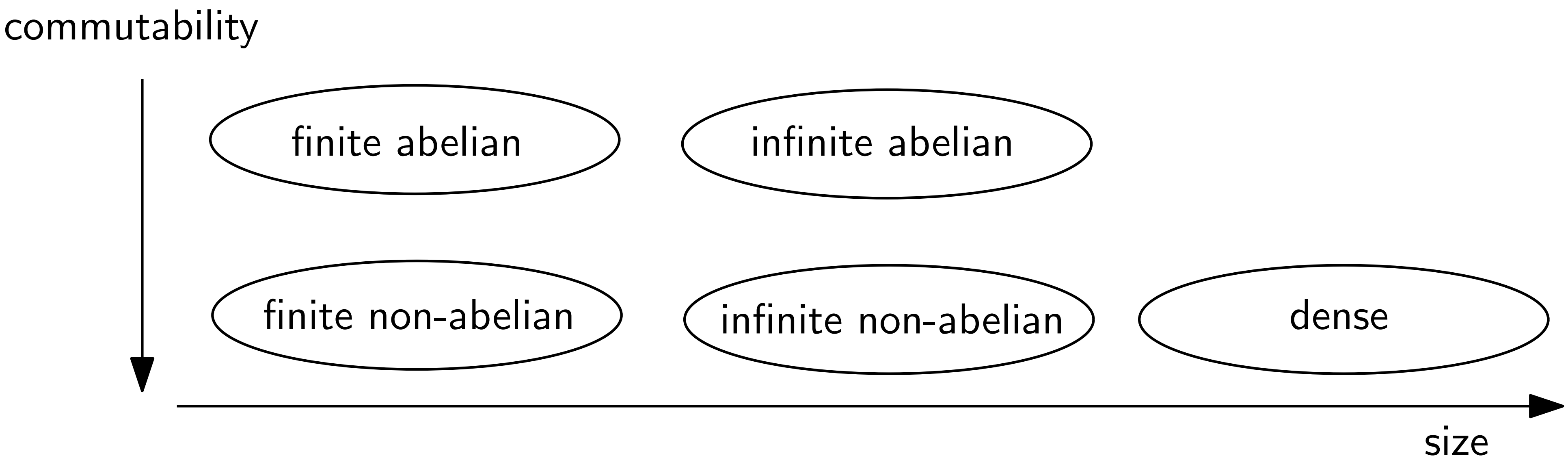
Lemma[Turaev, 2010; Barlett 2005]



$\overset{\text{TQFT}}{\Longrightarrow}$ {PSU matrices}

$$d-1 \text{ object } \Sigma \implies \text{representation } \pi : \text{Mod}(\Sigma) \to PSU(V)$$

The image of the representation Mod(genus 3 surface) induced by a 3-TQFT can be of five types in the groups $PSU(V)$



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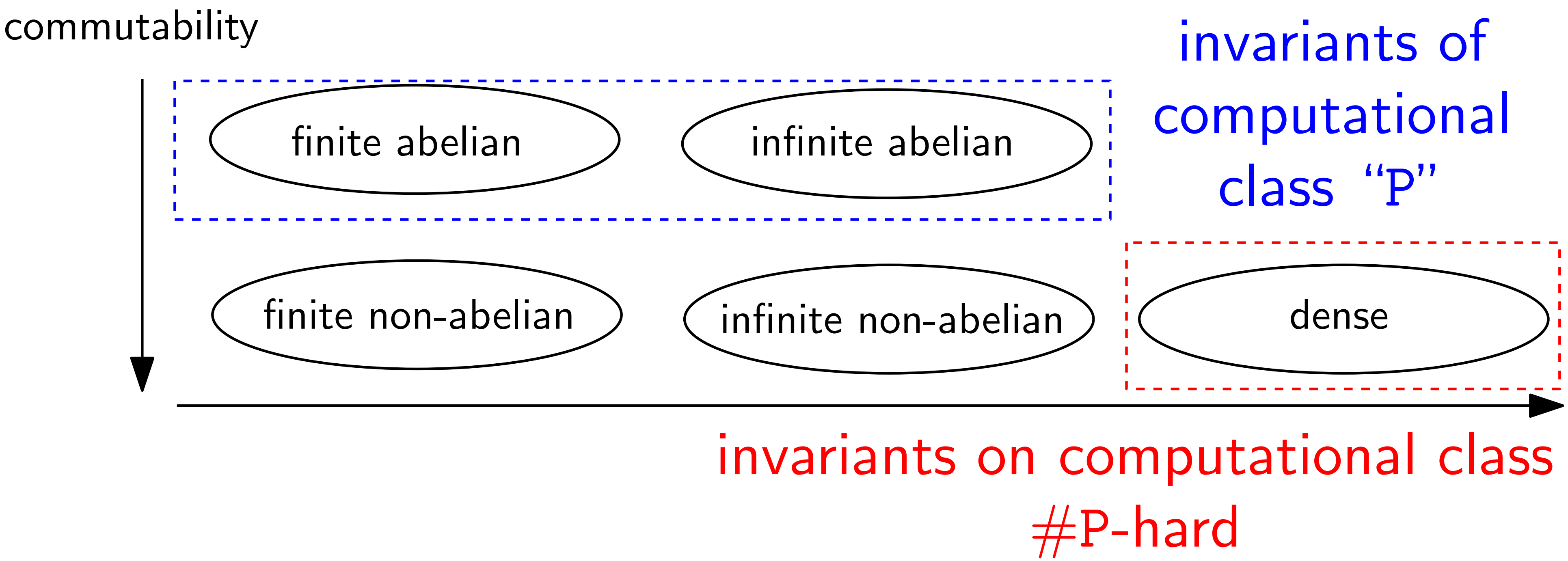

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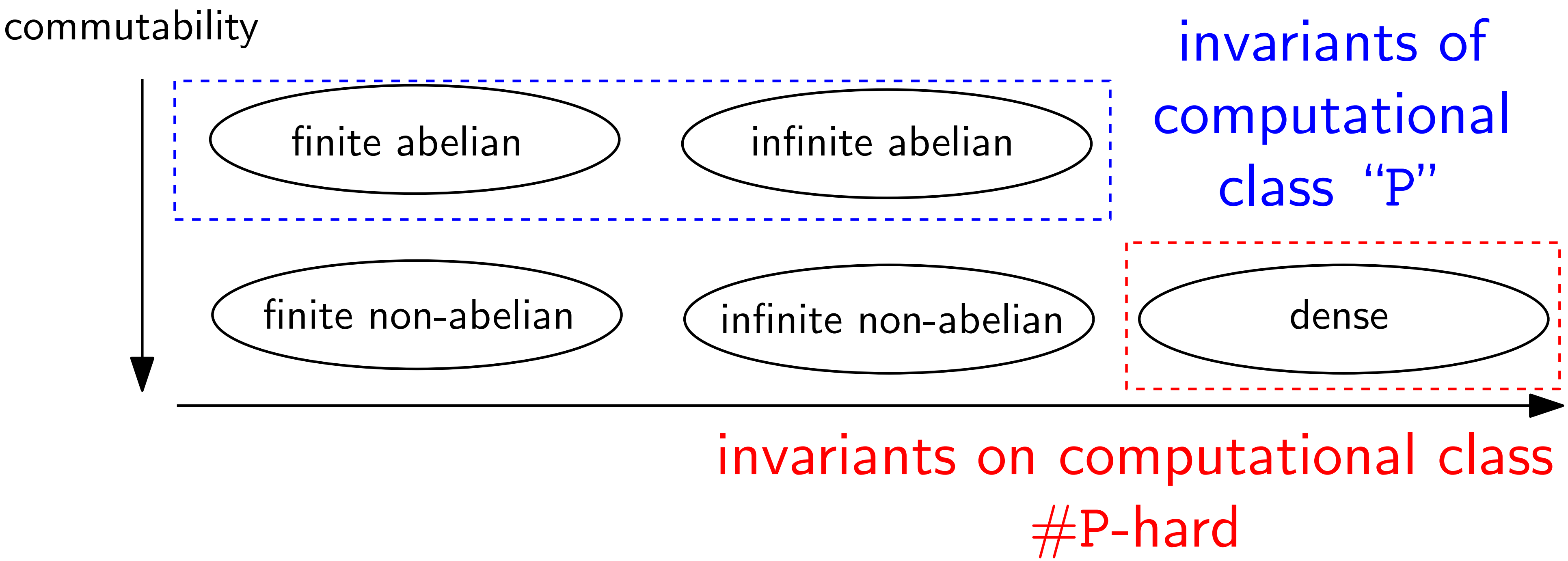

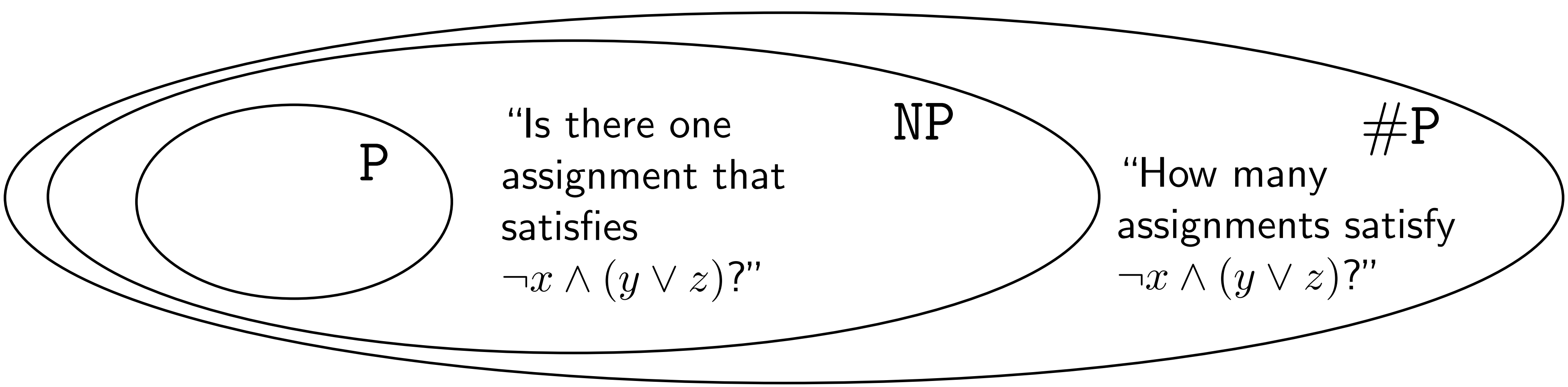

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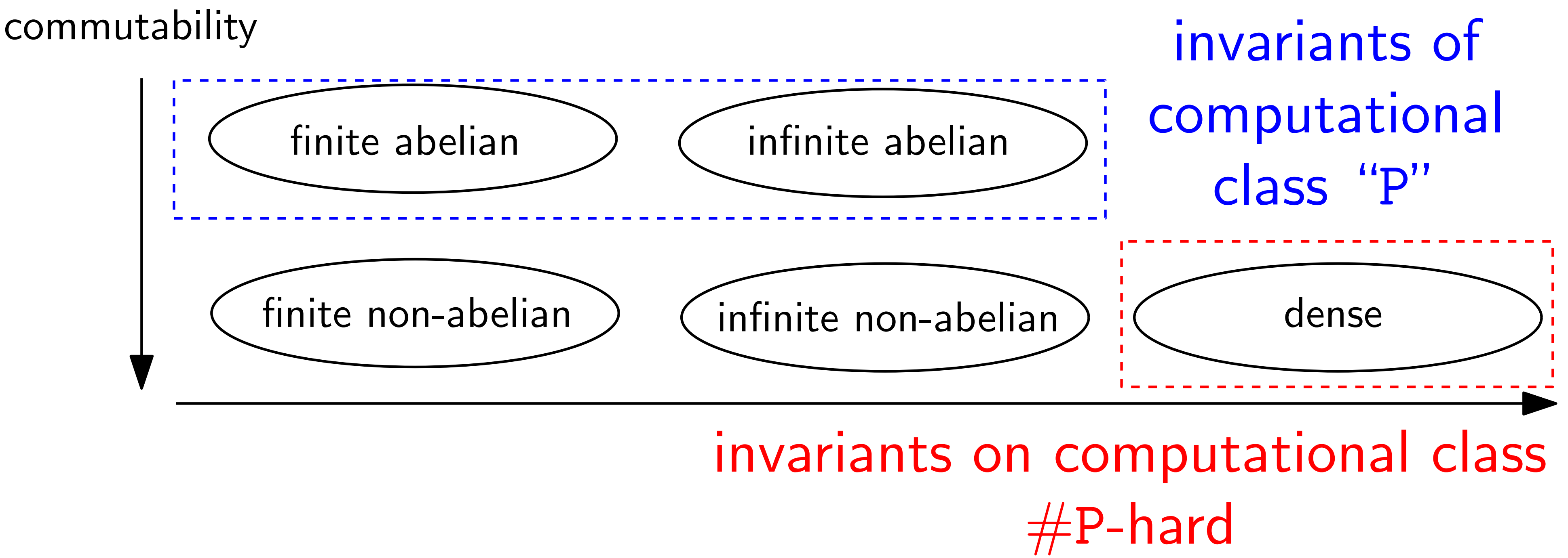

Lemma[Alagic and Lo, 2010. Theorem 3.2]: We can simulate *any* quantum computer using #P-hard invariants.

References

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